Transport in a Hollow Cylindrical Membrane

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SYNOPSIS

A detailed treatment has been given of radial transport in and through a hollow cylindrical membrane for two initial distributions of diffusant within the membrane of practical interest. Expressions are given for concentration distributions of diffusant and for unit length of membrane, amounts present, fluxes through radial planes, and amounts transported up to time t across those planes. Some consideration has been given to forward and reverse flow and to relations between flow quantities characteristic of those transports. © 1997 John Wiley & Sons, Inc.

INTRODUCTION

Hollow cylinders of uniform circular cross-section are an important type of membrane. Solutions for transport in such media are, in general, less well-documented than their counterparts for slab membranes and, moreover, are usually of more complex form. In an earlier note,[1] adsorption and desorption time-lags associated with purely *radial* transport under fixed (time-independent) boundary conditions were given for a homogeneous hollow circular cylinder and for a homogeneous spherical shell, both with constant diffusion coefficient D. In this note, corresponding transient state solutions for transport in and through a hollow circular cylinder are presented in order to provide a detailed source-reference for a number of initial and boundary conditions of practical interest to supplement cases already considered.[2,3]

The differential equation of transport for the hollow cylinder is:

$$\frac{\partial C}{\partial t} = \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\mathrm{D}\,\frac{\partial C}{\partial r}\right) = \mathrm{D}\left\{\frac{1}{r}\,\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial r^2}\right\} \quad (1)$$

and D is considered to be constant in what follows. With the initial and boundary conditions:

$$C(R_1, t) = C_1 = \text{constant}, \quad t > 0$$
$$C(R_2, t) = C_2 = \text{constant}, \quad t > 0$$
$$C(r, 0) = f(r), \quad R_1 < r < R_2 \quad (2)$$

the solution to eq. (1) is[2]

$$C(r, t) = \frac{C_1 \ln(R_2/r) + C_2 \ln(r/R_1)}{\ln(R_2/R_1)} - \pi \sum_{n=1}^{\infty} \frac{\{C_2 \mathrm{J_o}(\alpha_n R_1) - C_1 \mathrm{J_o}(\alpha_n R_2)\}}{\mathrm{J_o^2}(\alpha_n R_1) - \mathrm{J_o^2}(\alpha_n R_2)} \times \mathrm{J_o}(\alpha_n R_1) U_o(\alpha_n r) e^{-\alpha_n^2 \mathrm{D}t} + \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{\alpha_n^2 \mathrm{J_o^2}(\alpha_n R_1)\cdot U_o(\alpha_n r) e^{-\alpha_n^2 \mathrm{D}t}}{\mathrm{J_o^2}(\alpha_n R_1) - \mathrm{J_o^2}(\alpha_n R_2)} \times \int_{R_1}^{R_2} r\cdot f(r)\cdot U_o(\alpha_n r)\,\mathrm{d}r \quad (3)$$

where

$$U_o(\alpha r) \equiv \mathrm{J_o}(\alpha r)\mathrm{Y_o}(\alpha R_2) - \mathrm{J_o}(\alpha R_2)\mathrm{Y_o}(\alpha r) \quad (4)$$

and the α_n are the positive roots of

$$U_o(\alpha R_1) \equiv \mathrm{J_o}(\alpha R_1)\mathrm{Y_o}(\alpha R_2) - \mathrm{J_o}(\alpha R_2)\mathrm{Y_o}(\alpha R_1) = 0 \quad (5)$$

$\mathrm{J_o}$ and $\mathrm{Y_o}$ are zero-order Bessel functions of the first and second kind, respectively. The roots of eq. (5) are all real and simple and to each positive

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root there corresponds a negative root $(-)\ \alpha$.[2] A short table of these roots has been given by Carslaw and Jaeger[2] and is reproduced by Crank.[3]

In what follows, two special cases of the initial distribution $f(r)$ designated (A) and (B) are considered.

Case (A)

The concentration is initially uniform throughout the membrane. We write

$$f(r) = C_i = \text{constant}$$

so that from eq. A1

$$\int_{R_1}^{R_2} r \cdot f(r) U_o(\alpha_n r)\, dr = \frac{2C_i\{J_o(\alpha_n R_1) - J_o(\alpha_n R_2)\}}{\pi \alpha_n^2 J_o(\alpha_n R_1)} \quad (6)$$

and eq. (3) becomes

$$C(r,t) = \frac{C_1 \ln(R_2/r) + C_2 \ln(r/R_1)}{\ln(R_2/R_1)} - \pi \sum_{n=1}^{\infty} \frac{\{C_2 J_o(\alpha_n R_1) - C_1 J_o(\alpha_n R_2)\}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \times J_o(\alpha_n R_1) U_o(\alpha_n r) e^{-\alpha_n^2 Dt} + \pi C_i \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1) U_o(\alpha_n r) e^{-\alpha_n^2 Dt}}{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)} \quad (7)$$

as given by Crank.[3]

Case (B)

An initial steady-state distribution for the (fixed) boundary conditions

$$C(R_1, t) = C_1', t > 0$$

$$C(R_2, t) = C_2', t > 0$$

is established. Then

$$f(r) = \frac{C_1' \ln(R_2/r) + C_2' \ln(r/R_1)}{\ln(R_2/R_1)}$$

which is the steady-state distribution for the boundary concentrations C_1' and C_2' as can be seen by reference to eq. (3) with $t \to \infty$.

$$\int_{R_1}^{R_2} r f(r) U_o(\alpha_n r)\, dr = \frac{C_1' \ln(R_2) - C_2' \ln(R_1)}{\ln(R_2/R_1)} \int_{R_1}^{R_2} r \cdot U_o(\alpha_n r)\, dr - \frac{(C_1' - C_2')}{\ln(R_2/R_1)} \int_{R_1}^{R_2} r \cdot \ln(r) \cdot U_o(\alpha_n r)\, dr \quad (8)$$

which with eq. A(1) and A(2) reduces to

$$\int_{R_1}^{R_2} r f(r) U_o(\alpha_n r)\, dr = \frac{-2\{C_1' J_o(\alpha_n R_2) - C_2' J_o(\alpha_n R_1)\}}{\pi \alpha_n^2 J_o(\alpha_n R_1)} \quad (9)$$

and eq. (3) becomes

$$C(r,t) = \frac{C_1 \ln(R_2/r) + C_2 \ln(r/R_1)}{\ln(R_2/R_1)} - \pi \sum_{n=1}^{\infty} \frac{\{C_2 J_o(\alpha_n R_1) - C_1 J_o(\alpha_n R_2)\}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \times J_o(\alpha_n R_1) U_o(\alpha_n r) e^{-\alpha_n^2 Dt} + \pi \sum_{n=1}^{\infty} \frac{\{C_2' J_o(\alpha_n R_1) - C_1' J_o(\alpha_n R_2)\}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \times J_o(\alpha_n R_1) U_o(\alpha_n r) e^{-\alpha_n^2 Dt} \quad (10)$$

When $C_1' = C_2' = C_i$ eq. (10) reduces to equation (7).

AMOUNT OF DIFFUSANT, M_t, IN HOLLOW CYLINDER OF UNIT LENGTH

In general

$$M_t = \int_{R_1}^{R_2} 2\pi r \cdot C(r,t)\, dr \quad (11)$$

which with eq. A(1) and (3) yields

$$M_t = \pi(C_2 R_2^2 - C_1 R_1^2) + \frac{\pi(C_1 - C_2)(R_2^2 - R_1^2)}{2 \ln(R_2/R_1)} - 4\pi \sum_{n=1}^{\infty} \frac{\{C_2 J_o(\alpha_n R_1) - C_1 J_o(\alpha_n R_2)\} e^{-\alpha_n^2 Dt}}{\alpha_n^2 \{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)\}}$$

$$+ 2\pi^2 \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1) \int_{R_1}^{R_2} r f(r) U_o(\alpha_n r)\, dr e^{-\alpha_n^2 Dt}}{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)} \tag{12}$$

For the steady-state, $t \to \infty$, and

$$\lim_{t\to\infty} M_t \equiv M_\infty = \pi(C_2R_2^2 - C_1R_1^2) + \frac{\pi}{2}\frac{(C_1 - C_2)(R_2^2 - R_1^2)}{\ln(R_2/R_1)} \tag{13}$$

The initial amount in the membrane, M_i, is given by

$$M_i = \int_{R_1}^{R_2} 2\pi r f(r)\, dr \tag{14}$$

Considering the special cases we have:

Case (A)

Equation (14) becomes

$$M_i = \pi C_i (R_2^2 - R_1^2) \tag{15}$$

and eq. (12) and (6) give

$$M_t = \pi(C_2R_2^2 - C_1R_1^2) + \frac{\pi(C_1 - C_2)(R_2^2 - R_1^2)}{2\ln(R_2/R_1)} + 4\pi \sum_{n=1}^{\infty} \frac{\{(C_1 - C_i)J_o(\alpha_n R_2) - (C_2 - C_i)J_o(\alpha_n R_1)\}e^{-\alpha_n^2 Dt}}{\alpha_n^2\{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)\}} \tag{16}$$

The change in the amount of material in the membrane after a time t is given by $M_t - M_i$. The resultant expression has been given by Crank[3] who, however, denotes this quantity by M_t.

Case (B)

Eq. (14) becomes

$$M_i = \pi(C_2'R_2^2 - C_1'R_1^2) + \frac{\pi(C_1' - C_2')(R_2^2 - R_1^2)}{2\ln(R_2/R_1)} \tag{17}$$

and eq. (12) and (9) combine to give

$$M_t = \pi(C_2R_2^2 - C_1R_1^2) + \frac{\pi(C_1 - C_2)(R_2^2 - R_1^2)}{2\ln(R_2/R_1)} + 4\pi \sum_{n=1}^{\infty} \frac{\{(C_2' - C_2)J_o(\alpha_n R_1) - (C_1' - C_1)J_o(\alpha_n R_2)\}e^{-\alpha_n^2 Dt}}{\alpha_n^2\{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)\}} \tag{18}$$

FLUX, J(*r*, *t*), PER UNIT LENGTH OF CYLINDER THROUGH THE PLANE OF RADIUS *r*

This flux is given by

$$J(r, t) = -2\pi r D \frac{\partial C(r, t)}{\partial r} \tag{19}$$

which with eq. (3) and (A3) gives

$$J(r, t) = \frac{2\pi D(C_1 - C_2)}{\ln(R_2/R_1)} + 2\pi^2 D \times \sum_{n=1}^{\infty} \frac{\{C_2 J_o(\alpha_n R_1) - C_1 J_o(\alpha_n R_2)\} J_o(\alpha_n R_1)(\alpha_n r) U_o'(\alpha_n r) e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} - \pi^3 D \sum_{n=1}^{\infty} \frac{\alpha_n^2 J_o^2(\alpha_n R_1)\cdot(\alpha_n r) U_o'(\alpha_n r) e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \times \int_{R_1}^{R_2} r f(r) U_o(\alpha_n r)\, dr \tag{20}$$

The corresponding expressions for $J(R_1, t)$ and $J(R_2, t)$ follow in a straightforward fashion using the expressions for $U_o'(\alpha_n R_1)$ and $U_o'(\alpha_n R_2)$ given in the Appendix and for the special cases we have

Case (A)

From eq. (20) and (6)

$$J(r, t) = \frac{2\pi D(C_1 - C_2)}{\ln(R_2/R_1)} + 2\pi^2 D \times \sum_{n=1}^{\infty} \frac{\{(C_2 - C_i)J_o(\alpha_n R_1) - (C_1 - C_i)J_o(\alpha_n R_2)\}J_o(\alpha_n R_1) \times \alpha_n r U_o'(\alpha_n r) e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \tag{21}$$

and for

Case (B)

From eq. (20) and (9)

$$J(r,t) = \frac{2\pi D(C_1 - C_2)}{\ln(R_2/R_1)} + 2\pi^2 D \times \sum_{n=1}^{\infty} \frac{\{(C_2 - C_2')J_o(\alpha_n R_1) - (C_1 - C_1')J_o(\alpha_n R_2)\} J_o(\alpha_n R_1) \times (\alpha_n r) U_o'(\alpha_n r) e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \tag{22}$$

Again, the expressions for $J(R_1, t)$ and $J(R_2, t)$ follow readily from eq. (21), (22), (A4), and (A5).

Amount Transported, $Q(r, t)$, Across the Plane of Radius r

$$Q(r,t) = \int_o^t J(r,t)\,dt \equiv -\int_o^t 2\pi r D \frac{\partial C}{\partial r}\,dt \tag{23}$$

which with eq. (20) gives

$$Q(r,t) = \frac{2\pi D(C_1 - C_2)t}{\ln(R_2/R_1)} + 2\pi^2 \times \sum_{n=1}^{\infty} \frac{\{C_2 J_o(\alpha_n R_1) - C_1 J_o(\alpha_n R_2)\} J_o(\alpha_n R_1) \times (\alpha_n r) U_o'(\alpha_n r)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2 \{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)\}} - \pi^3 \sum_{n=1}^{\infty} \frac{J_o^2(\alpha_n R_1)\cdot(\alpha_n r) U_o'(\alpha_n r) \times \{1 - e^{-\alpha_n^2 Dt}\}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \times \int_{R_1}^{R_2} r f(r) U_o(\alpha_n r)\,dr \tag{24}$$

from which $Q(R_1, t)$ and $Q(R_2, t)$ follow using eq. (A4) and (A5), respectively.

For the special cases we have:

Case (A)

Eq. (24) and (6) give

$$Q(r,t) = \frac{2\pi D(C_1 - C_2)t}{\ln(R_2/R_1)} + 2\pi^2 \times \sum_{n=1}^{\infty} \frac{\{(C_2 - C_i)J_o(\alpha_n R_1) - (C_1 - C_i)J_o(\alpha_n R_2)\} J_o(\alpha_n R_1)(\alpha_n r) \times U_o'(\alpha_n r)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2 \{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)\}} \tag{25}$$

and for

Case B

From eq. (24) and (9)

$$Q(r,t) = \frac{2\pi D(C_1 - C_2)t}{\ln(R_2/R_1)} + 2\pi^2 \times \sum_{n=1}^{\infty} \frac{\{(C_2 - C_2')J_o(\alpha_n R_1) - (C_1 - C_1')J_o(\alpha_n R_2)\} J_o(\alpha_n R_1)(\alpha_n r) \times U_o'(\alpha_n r)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2 \{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)\}} \tag{26}$$

Again the corresponding quantities $Q(R_1, t)$ and $Q(R_2, t)$ follow readily.

CONSTRAINTS ON C_1 AND C_2

At this point no constraint has been placed on the relative magnitudes of C_1 and C_2. There are two possibilities.

a) $C_1 = C_2$: constant pressure sorption/desorption kinetics

b) $C_1 \neq C_2$: transport through the cylinder wall

We now consider in turn the modifications to the expressions for $C(r, t)$ [eq. (7)], M_t [eq. (16)], $J(r, t)$ [eq. (21)] and $Q(r, t)$ [eq. (25)] for the two initial distributions [case (A) and case (B)] when the constraints (a) and (b) are imposed.

a) $C_1 = C_2 = C_0$

For CASE (A), Eq. (7) becomes

$$C(r,t) = C_0 - \pi(C_0 - C_i) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1) U_o(\alpha_n r) e^{-\alpha_n^2 Dt}}{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)}, \tag{27}$$

Eq. (16) becomes

$$M_t = \pi C_o(R_2^2 - R_1^2) - 4\pi(C_o - C_i) \times \sum_{n=1}^{\infty} \frac{\{J_o(\alpha_n R_1) - J_o(\alpha_n R_2)\}e^{-\alpha_n^2 Dt}}{\alpha_n^2\{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)\}} \quad (28)$$

and

$$\frac{M_t - M_i}{M_\infty - M_i} = 1 - \frac{4}{(R_2^2 - R_1^2)} \times \sum_{n=1}^{\infty} \frac{\{J_o(\alpha_n R_1) - J_o(\alpha_n R_2)\}e^{-\alpha_n^2 Dt}}{\alpha_n^2\{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)\}} \quad (29)$$

Also, eq. (21) becomes

$$J(r, t) = 2\pi^2 D(C_o - C_i) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1)(\alpha_n r)U_o'(\alpha_n r)e^{-\alpha_n^2 Dt}}{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)} \quad (30)$$

which with eq. A4 and A5 gives

$$J(R_2, t) = -4\pi D(C_o - C_i) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1)e^{-\alpha_n^2 Dt}}{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)} \quad (31)$$

and

$$J(R_1, t) = -4\pi D(C_o - C_i) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_2)e^{-\alpha_n^2 Dt}}{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)} \quad (32)$$

Eq. (25) becomes

$$Q(r, t) = 2\pi^2(C_o - C_i) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1)(\alpha_n r)U_o'(\alpha_n r)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2\{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)\}} \quad (33)$$

so that

$$Q(R_2, t) = -4\pi(C_o - C_i) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2\{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)\}} \quad (34)$$

and

$$Q(R_1, t) = -4\pi(C_o - C_i) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_2)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2\{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)\}} \quad (35)$$

For CASE (B), eq. (10) becomes

$$C(r, t) = C_o - \pi \times \sum_{n=1}^{\infty} \frac{\{(C_o - C_2')J_o(\alpha_n R_1) - (C_o - C_1')J_o(\alpha_n R_2)\}J_o(\alpha_n R_1) \times U_o(\alpha_n r)e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \quad (36)$$

Eq. (18) becomes

$$M_t = \pi C_o(R_2^2 - R_1^2) - 4\pi \times \sum_{n=1}^{\infty} \frac{\{(C_o - C_2')J_o(\alpha_n R_1) - (C_o - C_1')J_o(\alpha_n R_2)\}e^{-\alpha_n^2 Dt}}{\alpha_n^2\{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)\}} \quad (37)$$

and eq. (22) becomes

$$J(r, t) = 2\pi^2 D \times \sum_{n=1}^{\infty} \frac{\{(C_o - C_2')J_o(\alpha_n R_1) - (C_o - C_1')J_o(\alpha_n R_2)\}J_o(\alpha_n R_1)(\alpha_n r) \times U_o'(\alpha_n r)e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \quad (38)$$

Then,

$$J(R_2, t) = -4\pi D \times \sum_{n=1}^{\infty} \frac{\{(C_o - C_2')J_o(\alpha_n R_1) - (C_o - C_1')J_o(\alpha_n R_2)\}J_o(\alpha_n R_1)e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \quad (39)$$

and

$$J(R_1, t) = -4\pi D \times \sum_{n=1}^{\infty} \frac{\{(C_o - C_2')J_o(\alpha_n R_1) - (C_o - C_1')J_o(\alpha_n R_2)\}J_o(\alpha_n R_2)e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \quad (40)$$

Also,

$$Q(r, t) = 2\pi^2 \times \sum_{n=1}^{\infty} \frac{\{(C_o - C_2')J_o(\alpha_n R_1) - (C_o - C_1')J_o(\alpha_n R_2)\}J_o(\alpha_n R_1)(\alpha_n r) \times U_o'(\alpha_n r)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2\{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)\}} \quad (41)$$

so that

$$Q(R_2, t) = -4\pi \times \sum_{n=1}^{\infty} \frac{\{(C_o - C_2')J_o(\alpha_n R_1) - (C_o - C_1')J_o(\alpha_n R_2)\}J_o(\alpha_n R_1) \times \{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2\{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)\}} \quad (42)$$

and

$$Q(R_1, t) = -4\pi \times \sum_{n=1}^{\infty} \frac{\{(C_o - C_2')J_o(\alpha_n R_1) - (C_o - C_1')J_o(\alpha_n R_2)\}J_o(\alpha_n R_2) \times \{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2\{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)\}} \quad (43)$$

For the constraint,
(b) $C_1 \neq C_2$:
all previous equations containing C_1 and C_2 apply. We need to consider $C_1 > C_2$ and $C_1 < C_2$, but in so doing we restrict our considerations to $C_i = C_1$ and $C_i = C_2$ for CASE (A) only, i.e., $(f(r) = C_i)$.

The following notation is introduced:

$C_1 > C_2$ defines "forward" flow (in direction of r increasing)

$C_1 < C_2$ defines "reverse" flow (in direction of r decreasing)

and the following boundary conditions considered:

(I) $C_1 \neq C_2 = C_i$

(II) $C_2 \neq C_1 = C_i$

System (I) includes the cases of forward adsorption flow ($C_1 > C_2 = C_i$) and reverse, desorption flow ($C_1 < C_2 = C_i$). The expressions given below are for forward adsorption flows. The corresponding expressions for reverse, desorption flows are obtained by interchanging C_1 and C_2.

From eq. (7)

$$C(r, t) = \frac{C_1 \ln(R_2/r) + C_2 \ln(r/R_1)}{\ln(R_2/R_1)} + \pi(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1)J_o(\alpha_n R_2)U_o(\alpha_n r)e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \quad (44)$$

From eq. (16)

$$M_t = \pi(C_2 R_2^2 - C_1 R_1^2) + \frac{\pi(C_1 - C_2)(R_2^2 - R_1^2)}{2\ln(R_2/R_1)} + 4\pi \times \sum_{n=1}^{\infty} \frac{(C_1 - C_2)J_o(\alpha_n R_2)e^{-\alpha_n^2 Dt}}{\alpha_n^2\{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)\}} \quad (45)$$

From eq. (21)

$$J(r, t) = \frac{2\pi D(C_1 - C_2)}{\ln(R_2/R_1)} - 2\pi^2 D(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1)J_o(\alpha_n R_2) \times (\alpha_n r)U_o'(\alpha_n r)e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \quad (46)$$

so that

$$J(R_2, t) = \frac{2\pi D(C_1 - C_2)}{\ln(R_2/R_1)} + 4\pi D(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1)J_o(\alpha_n R_2)e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \quad (47)$$

and

$$J(R_1, t) = \frac{2\pi D(C_1 - C_2)}{\ln(R_2/R_1)} + 4\pi D(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o^2(\alpha_n R_2)e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \quad (48)$$

From eq. (25)

$$Q(r, t) = \frac{2\pi D(C_1 - C_2)t}{\ln(R_2/R_1)} - 2\pi^2(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1)J_o(\alpha_n R_2) \times (\alpha_n r)U_o'(\alpha_n r)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2\{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)\}} \quad (49)$$

$$Q(R_2, t) = \frac{2\pi D(C_1 - C_2)t}{\ln(R_2/R_1)} + 4\pi(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1)J_o(\alpha_n R_2)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2\{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)\}} \quad (50)$$

and

$$Q(R_1, t) = \frac{2\pi D(C_1 - C_2)t}{\ln(R_2/R_1)} + 4\pi(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o^2(\alpha_n R_2)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2\{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)\}} \quad (51)$$

System (II) includes the cases of forward, desorption flow ($C_i = C_1 > C_2$) and reverse, adsorption flow ($C_i = C_1 < C_2$). The expressions given below are for forward, desorption flows. The corresponding expressions for reverse, adsorption flows are obtained by interchanging C_1 and C_2.

As for system (I) we now have,

$$C(r,t) = \frac{C_1 \ln(R_2/r) + C_2 \ln(r/R_1)}{\ln(R_2/R_1)} + \pi(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o^2(\alpha_n R_1) U_o(\alpha_n r) e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \quad (52)$$

$$M_t = \pi(C_2 R_2^2 - C_1 R_1^2) + \frac{\pi(C_1 - C_2)(R_2^2 - R_1^2)}{2 \ln(R_2/R_1)} + 4\pi(C_1 - C_2) \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1) e^{-\alpha_n^2 Dt}}{\alpha_n^2 \{J_o(\alpha_n R_1) + J_o(\alpha_n R_2)\}} \quad (53)$$

$$J(r,t) = \frac{2\pi D(C_1 - C_2)}{\ln(R_2/R_1)} - 2\pi^2 D(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o^2(\alpha_n R_1)(\alpha_n r) U_o'(\alpha_n r) e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \quad (54)$$

$$J(R_2,t) = \frac{2\pi D(C_1 - C_2)}{\ln(R_2/R_1)} + 4\pi D(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o^2(\alpha_n R_1) e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \quad (55)$$

$$J(R_1,t) = \frac{2\pi D(C_1 - C_2)}{\ln(R_2/R_1)} + 4\pi D(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1) J_o(\alpha_n R_2) e^{-\alpha_n^2 Dt}}{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)} \quad (56)$$

$$Q(r,t) = \frac{2\pi D(C_1 - C_2)t}{\ln(R_2/R_1)} - 2\pi^2 (C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o^2(\alpha_n R_1)(\alpha_n r) U_o'(\alpha_n r)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2 \{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)\}} \quad (57)$$

$$Q(R_2,t) = \frac{2\pi D(C_1 - C_2)t}{\ln(R_2/R_1)} + 4\pi(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o^2(\alpha_n R_1)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2 \{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)\}} \quad (58)$$

$$Q(R_1,t) = \frac{2\pi D(C_1 - C_2)t}{\ln(R_2/R_1)} + 4\pi(C_1 - C_2) \times \sum_{n=1}^{\infty} \frac{J_o(\alpha_n R_1) J_o(\alpha_n R_2)\{1 - e^{-\alpha_n^2 Dt}\}}{\alpha_n^2 \{J_o^2(\alpha_n R_1) - J_o^2(\alpha_n R_2)\}} \quad (59)$$

Relations Between Forward and Reverse Flow Quantities[1]

Subscripts a and d denote "adsorption" and "desorption," respectively, and the superscript * denotes "reverse" flow.

From eq. (44) to (59) we have:

$$C_a(r,t) + C_d^*(r,t) = C_d(r,t) + C_a^*(r,t) = C_1 + C_2 \quad (60)$$

$$(M_t)_a + (M_t^*)_d = (M_t)_d + (M_t^*)_a = \pi(R_2^2 - R_1^2)(C_1 + C_2) \quad (61)$$

$$J_a(r,t) = -J_d^*(r,t) \quad (62)$$

$$J_d(r,t) = -J_a^*(r,t) \quad (63)$$

$$Q_a(r,t) = -Q_d^*(r,t) \quad (64)$$

$$Q_d(r,t) = -Q_a^*(r,t) \quad (65)$$

Finally,

$$J_a(R_2,t) = -J_d^*(R_2,t) = J_d(R_1,t) = -J_a^*(R_1,t) \quad (66)$$

and so

$$Q_a(R_2,t) = -Q_d^*(R_2,t) = Q_d(R_1,t) = -Q_a^*(R_1,t) \quad (67)$$

The time lags for diffusion pertaining to the quantities of eq. (66) and (67) have been discussed elsewhere.[1]

APPENDIX

From eq. (4) we have

$$U_o(\alpha_n r) = J_o(\alpha_n r) Y_o(\alpha_n R_2) - J_o(\alpha_n R_2) Y_o(\alpha_n r)$$

Then,[2]

a) $$\int_{R_1}^{R_2} rU_o(\alpha_n r)\,dr = \frac{2\{J_o(\alpha_n R_1) - J_o(\alpha_n R_2)\}}{\pi\alpha_n^2 J_o(\alpha_n R_1)} \quad (A1)$$

b) $$\int_{R_1}^{R_2} r \cdot \ln(r)(U_o(\alpha_n r)\,dr = \frac{2\{J_o(\alpha_n R_1)\ln(R_2) - J_o(\alpha_n R_2)\ln(R_1)\}}{\pi\alpha_n^2 J_o(\alpha_n R_1)} \quad (A2)$$

c) $$U_o'(\alpha_n r) = \frac{1}{\alpha_n}\frac{d}{dr}U_o(\alpha_n r) \quad (A3)$$

giving

d) $$U_o'(\alpha_n R_2) = -2/\pi\alpha_n R_2 \quad (A4)$$

e) $$U_o'(\alpha_n R_1) = -\frac{2}{\pi\alpha_n R_1}\cdot\frac{J_o(\alpha_n R_2)}{J_o(\alpha_n R_1)} \quad (A5)$$

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